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by Naimah Aris Et Al

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On Weak Solution of Cauchy Problem for Nonlinear Parabolic Equations

Naimah Aris¹, Muh. Rifki Nisardi¹, Kasbawati¹, Jeffry Kusuma¹, Budi Nurwahyu¹, Mawardi¹

¹Department of Mathematics, Hasanuddin University, Makassar, Indonesia

E-mail: newima@gmail.com

Abstract. In this paper, we investigate weak solutions of Cauchy problem for nonlinear parabolic equations. By using Galerkin approximation, the existence and uniqueness of weak solutions are proved.

1. Introduction

There are several problems in partial differential equations, of which classical solution is difficult to solve. Different approaches are needed to ensure the existence of a solution of the equations. One way to determine the solution of a differential equation is to expand the space of solution which contains the smooth solution and the non-smooth solution. The solution contained in this space is called a weak solution, [3].

The existence of the weak solution of nonlinear parabolic equations has been investigated in many papers. Qiang *et al.*, [2] have shown the existence and some properties of the weak solution of nonlinear parabolic equations without formulating it into abstract Cauchy problem.

In partial differential equations theory, some mathematical problems endowed by appropriate conditions and spaces can be transformed into an abstract Cauchy problem. Many papers deal with the Cauchy problem for nonlinear parabolic equations, [2-8]. Choe and Lee [4] explain the existence and uniqueness of Cauchy problem for degenerate parabolic equations using Harnack estimate. Shang [6] study the existence and non-existence of local and global solutions of the Cauchy problem by a priori estimate and compactness methods. The properties of the solutions of the Cauchy problem for non-divergent form with density, appear in Rainbekov's paper in 2015.

In this paper, we discuss the weak solution of the abstract Cauchy problem of nonlinear parabolic equations

$$\begin{aligned} \frac{\partial u}{\partial t} + Au &= f(u) + \delta \\ u(0) &= u_0 \end{aligned} \quad (1)$$

where A is a continuous linear operator that does not depend on time, $A = -\Delta$ and f is a continuous function. We use Galerkin approximation [1] to show the existence of the weak solution. For the proof, we use the following notation

$$\begin{aligned} \Psi &= \{u \in \mathcal{D}(\Omega), \operatorname{div} u = 0\} \\ V &= \text{Closure from } \Psi \text{ in } H_0^1(\Omega) \end{aligned}$$



$$H = \text{Closure from } \Psi \text{ in } L^2(\Omega)$$

Let V and H be Hilbert spaces. $V \subset H$, V dense in H with injective dan continuous function. Scalar product and norm of V and H are denoted by (\cdot, \cdot) , $\|\cdot\|$, (\cdot, \cdot) , $|\cdot|$. H is recognizable using its dual, H' which can be identified using by a dense subspace V , hence

$$V \subset H \subset V'$$

Scalar product in of $f \in H$ and $u \in V$ given by

$$\langle f, u \rangle = (f, u), \forall f \in H, \forall u \in V. \quad (2)$$

For every u in V

$$v \in V \rightarrow ((u, v)) \in \mathbb{R}$$

be a linear and continuous map in V . There exist an element in V' denote by Au such that

$$\langle Au, v \rangle = ((u, v)), \forall v \in V. \quad (3)$$

where $A: V \rightarrow V'$ be a linear and continuous operator. $A \in \mathcal{L}(V, V')$ define on V , so that A is a canonic isomorfism of V on V' .

Assuming that u is the solution to equation (1), with $u \in C^2(\Omega)$. If v is element of Ψ , we obtain

$$\left(\frac{du}{dt}, v\right) + ((u, v)) = (f, v) + (\delta, v).$$

From the continuity of equation (3), for each $v \in V$ apply

$$\left(\frac{du}{dt}, v\right) = \frac{d}{dt}(u, v).$$

It is the basis to form a weak solution to a problem in the equation (1). For $f \in L^2(0, T; V')$ and the given $u_0 \in H$, so will be determined u that fulfil

$$u \in L^2(0, T; V)$$

and

$$\frac{d}{dt}(u, v) + ((u, v)) = \langle f, v \rangle + \langle \delta, v \rangle, \forall v \in V.$$

Then the problem (1) can be formulated as follows

$$u' + Au = f(u) + \delta \text{ untuk } t \in (0, T),$$

$$u(0) = u_0.$$

Equation (2) is an equation distributed on V' . Based on (2), Equation (3) can be formulated as

$$\begin{aligned} u' &= (f(u) + \delta) - Au \\ \frac{d}{dt}(u, v) &= \langle (f(u) + \delta) - Au, v \rangle, \forall v \in V. \end{aligned} \quad (4)$$

Since A a linear operator and continuous from V to V' and $u \in L^2(0, T; V)$, so that function Au is found on $L^2(0, T; V')$. Hence, $(f(u) + \delta) - Au \in L^2(0, T; V')$ and from Equation (4) as well as Lemma 1 indicate that

$$u' \in L^2(0, T; V').$$

In this case, a function should determine $u \in L^2(0, T; V) \cap C^0(0, T; H)$ in such a way that

$$u_t = \frac{du}{dt} \in L^2(0, T; V').$$

Lemma 1

Let Ω be an open bounded set in \mathbb{R}^m , and g_j is a sequence function on $L^p(\Omega)$ with

$$\|g_j\|_{L^p(\Omega)} \leq C \quad \text{for each } j \in \mathbb{Z}^+$$

If $g \in L^p(\Omega)$ and $g_j \rightarrow g$ almost in every point on $g_j \rightarrow g$ di $L^p(\Omega)$.

Theorem 1

Let $X \subset H \subset Y$ is Banach space, with X is reflexive space. Given that u_n is a limited uniform term on $L^2(0, T; X)$ and $\frac{du_n}{dt}$ limited uniform on $L^p(0, T; Y)$, for $p > 1$. Then, a strong convergent sub term exists on $L^2(0, T; H)$.

Lemma 2

Let X is a reflexive Banach space and x_n is a limited-term X . Then, x_n has a weak convergent sub term on X .

The following conditions will use throughout the paper

Assumption 1

f is a function of $L^2(0, T; V')$ fulfilling

$$-k - \alpha_1 |s|^p \leq f(s) \leq k - \alpha_2 |s|^p, \quad p > 2 \quad (5)$$

$$f'(s) \leq l. \quad (6)$$

2. Result

Equation (1) fulfil an equation at $L^q(0, T; H^{-s}(\Omega))$, where q is conjugate pair at equation (5). Also (1) fulfil similarity at $L^q(0, T; H^{-s}(\Omega))$, hence for any $v \in L^p(0, T; H^s(\Omega))$ we have

$$\left\langle \frac{du}{dt}, v \right\rangle + \langle Au, v \rangle = \langle f(u), v \rangle + \langle \delta, v \rangle$$

$$\left\langle \frac{du}{dt}, v \right\rangle + \langle (u, v) \rangle = \langle f(u), v \rangle + \langle \delta, v \rangle$$

for almost each $t \in [0, T]$.

Using Sobolev embedding theorem, obtained rules

$$s \geq m(p - 2)/2p$$

given p dependent on function given on equation (5). If chosen $m = 1$ or 2 , then obtained $s = 1$, so that the equation obtained in familiar space $L^q(0, T; H^{-1}(\Omega))$ or $L^q(0, T; V^*(\Omega))$.

Theorem 2 (Weak Solution)

Equation (1) and f is a function of C^1 fulfilling by assumption A, have a unique weak solution u for any $T > 0$ and given $u_0 \in H$ with

$$u \in L^2(0, T; H_0^1(\Omega) \cap L^p(\Omega_T)) \text{ And } u \in C^0([0, T]; L^2(\Omega))$$

in addition, $u_0 \mapsto u(t)$ continuous on $L^2(\Omega)$.

Proof:

First, assume that V separated. Form independent linear sequence of complete elements V ; $\omega_1, \dots, \omega_m, \dots$. Observe for any value of m can be defined as a form of approach solution u_m for Equation (1) defined as follows

$$u_m(t) = \sum_{i=1}^n g_{im} \omega_i.$$

Function g_{im} , $1 \leq i \leq m$ is a function of scalar defined in $[0, T]$, and multiplication result of ω_j , Equation (1) and u_m obtained

$$\begin{aligned} \left\langle \frac{du_m}{dt} + Au_m, \omega_j \right\rangle &= \langle f(u) + \delta, \omega_j \rangle \\ \left\langle \frac{du_m}{dt}, \omega_j \right\rangle + \langle Au_m, \omega_j \rangle &= \langle f(u), \omega_j \rangle + \langle \delta, \omega_j \rangle \\ \langle u_m', \omega_j \rangle + \left((u_m, \omega_j) \right)' &= \langle f(u), \omega_j \rangle + \langle \delta, \omega_j \rangle \\ u_m(0) &= u_{0m} \quad \omega_j \in V. \end{aligned}$$

u_{0m} is orthogonal projection on H from u_0 in space spanned by $\omega_1, \dots, \omega_n$.

Multiply Equation (7) with $g_{jm}(t)$ so that obtained

$$\langle u_m', \omega_j \rangle g_{jm}(t) + \left((u_m, \omega_j) \right)' g_{jm}(t) = \langle f(u_m), \omega_j \rangle g_{jm}(t) + \langle \delta, \omega_j \rangle g_{jm}(t).$$

Then sum both term obtained

$$\sum \langle u_m', \omega_j \rangle g_{jm}(t) + \sum \left((u_m, \omega_j) \right)' g_{jm}(t) = \sum \langle f(u_m), \omega_j \rangle g_{jm}(t) + \sum \langle \delta, \omega_j \rangle g_{jm}(t).$$

Since $\sum \omega_j g_{jm}(t) = \sum g_{jm}(t) \omega_j = u_m(t)$, we have

$$\langle u_m'(t), u_m \rangle + \left((u_m, u_m) \right)' = \langle f(u_m), u_m \rangle + \langle \delta, u_m \rangle.$$

Since

$$2(u_m'(t), u_m(t)) = \frac{d}{dt} |u_m(t)|^2$$

we obtain

$$\frac{1}{2} \frac{d}{dt} |u_m|^2 + \|u_m\|^2 = \int_{\Omega} f(u_m) u_m dx + \int_{\Omega} \delta u_m dx.$$

Using the Equation (5), we obtain

$$\frac{1}{2} \frac{d}{dt} |u_m|^2 + \|u_m\|^2 \leq \int_{\Omega} k - \alpha_2 |u_m|^p dx + \delta \int_{\Omega} u_m dx$$

$$\frac{1}{2} \frac{d}{dt} |u_m|^2 + \|u_m\|^2 \leq \int_{\Omega} k - \alpha_2 |u_m|^p dx + |\delta| \|u_m\|.$$

Using Young inequality $ab \leq \frac{1}{2}(a^2 + b^2)$

$$\frac{1}{2} \frac{d}{dt} |u_m|^2 + \|u_m\|^2 \leq \int_{\Omega} k - \alpha_2 |u_m|^p dx + \frac{1}{2} (|\delta|^2 + \|u_m\|^2)$$

$$\frac{1}{2} \frac{d}{dt} |u_m|^2 + \|u_m\|^2 \leq \int_{\Omega} k - \alpha_2 |u_m|^p dx + \frac{1}{2} |\delta|^2 + \frac{1}{2} \|u_m\|^2$$

$$\frac{1}{2} \frac{d}{dt} |u_m|^2 + \frac{1}{2} \|u_m\|^2 \leq \int_{\Omega} k - \alpha_2 |u_m|^p dx + \frac{1}{2} |\delta|^2.$$

Integrate both term from 0 to T

$$\begin{aligned} & \frac{1}{2} \int_0^T \frac{d}{dt} |u_m(t)|^2 dt + \frac{1}{2} \int_0^T \|u_m(t)\|^2 dt \\ & \leq \int_0^T \int_{\Omega} k dx dt - \int_0^T \int_{\Omega} \alpha_2 |u_m(t)|^p dx dt + \int_0^T \frac{1}{2} |\delta|^2 dt \\ \frac{1}{2} |u_m(T)|^2 - \frac{1}{2} |u_m(0)|^2 + \frac{1}{2} \int_0^T \|u_m(t)\|^2 dt + \int_0^T \int_{\Omega} \alpha_2 |u_m(x,t)|^p dx dt & \leq kT|\Omega| + \frac{1}{2} |\delta|^2 T \\ \frac{1}{2} |u_m(T)|^2 + \frac{1}{2} \int_0^T \|u_m(t)\|^2 dt + \int_0^T \int_{\Omega} \alpha_2 |u_m(x,t)|^p dx dt & \leq \frac{1}{2} |u_0|^2 + kT|\Omega| + \frac{1}{2} |\delta|^2 T. \end{aligned}$$

Given $|\Omega|$ is the measurement of Ω , $|\Omega| = \int_{\Omega} dx$. Based on the above inequality, we obtain

$$\begin{aligned} \sup |u_m(T)|^2 & \leq 2\gamma \\ \int_0^T \|u_m(t)\|^2 & \leq 2\gamma \\ \int_0^T \int_{\Omega} |u_m(x,t)|^p dx dt & \leq \gamma/\alpha_2 \end{aligned}$$

where $\gamma = \frac{1}{2} |u_0|^2 + kT|\Omega| + \frac{1}{2} |\delta|^2 T$, hence, based on the boundary, we can write

$$u_m \text{ bounded uniform on } L^{\infty}(0, T; H).$$

$$u_m \text{ bounded uniform on } L^2(0, T; V).$$

$$u_m \text{ bounded uniform on } L^p(\Omega_T).$$

Sequence u_m that is bounded uniform on $L^p(\Omega_T)$ can be used to obtain the limit on a nonlinear function of $f(u_m)$ since

$$|f(s)| \leq \beta(|s|^{p-1} + 1).$$

From Equation (5), boundary on u_m di $L^p(\Omega_T)$ can be used to obtain the boundary of $f(u_m)$ on $L^q(\Omega_T)$ with (p, q) is a conjugate pair, hence

$$\begin{aligned} \|f(u_m)\|_{L^q(\Omega_T)}^q & = \int_0^T \left(\int_{\Omega} |f(u_m)|^q dx \right) dt \\ \|f(u_m)\|_{L^q(\Omega_T)}^q & \leq \beta \int_0^T \left(\left(\int_{\Omega} |u_m|^{p-1} + 1 \right)^q dx \right) dt \\ \|f(u_m)\|_{L^q(\Omega_T)}^q & \leq C \int_0^T \left(\int_{\Omega} |u_m|^{q(p-1)} + 1 dx \right) dt. \end{aligned}$$

Since $\frac{1}{p} + \frac{1}{q} = 1$, such that $q(p-1) = p$, then

$f(u_m)$ bounded uniform on $L^q(\Omega_T)$.

Furthermore, bounded uniform is required to derive equation (1). It is worth noted that both spaces $L^2(0, T; V^*)$ and $L^q(0, T; L^q(\Omega))$ contain continue on $L^q(0, T; H^{-s}(\Omega))$, with $q < 2$ karena $p > 2$. Then use (4)

$$\frac{du_m}{dt} = -Au_m + P_m f(u_m) + \delta$$

So that $\frac{du_m}{dt}$ bounded uniform on $L^q(0, T; H^{-s}(\Omega))$.

Using Lemma 2 to derive a sub term with weak convergence, namely u_m

$$u_m \rightharpoonup u \text{ on } L^2(0, T; V)$$

$$u_m \rightharpoonup u \text{ on } L^p(\Omega_T)$$

$$f(u_m) \rightharpoonup \chi \text{ on } L^q(\Omega_T).$$

Applying compactness property on Theorem 1, an additional sub term can be defined as

$$u_m \rightarrow u \text{ di } L^2(0, T; H). \quad (8)$$

It has been revealed previously that u_m bounded uniform on $L^2(0, T; H_0^1(\Omega))$ and also $\frac{du_m}{dt}$ bounded uniform on $L^q(0, T; H^{-s}(\Omega))$. Since $H_0^1(\Omega) \subset \subset L^2(\Omega) \subset H^{-s}(\Omega)$ and $H_0^1(\Omega)$ is Hilbert space, so $H_0^1(\Omega)$ is reflexive. Therefore, we use compactness theorem to acquire (8)

In the next step, we prove the convergence of $P_m f(u_m) \rightharpoonup \chi \text{ di } L^q(\Omega_T)$.

$$\int_{\Omega_T} (P_m f(u_m) - \chi) \phi dx dt = \int_{\Omega_T} (f(u_m) - \chi) \phi dx dt - \int_{\Omega_T} Q_m f(u_m) \phi dx dt$$

for any given $\phi \in L^p(\Omega_T)$. It has been revealed that the first part of the right term close to 0. For the second part of the right term, we define function ϕ as

$$\phi = \sum_{j=1}^m \alpha_j(t) \phi_j$$

Let $\alpha_j \in L^p(0, T)$, $\phi_j \in C_c^\infty(\Omega)$ dense on $L^p(\Omega_T)$. So for function

$$\int_{\Omega_T} Q_m f(u_m) \sum_{j=1}^m \alpha_j(t) \phi_j dx dt = \int_{\Omega_T} f(u_m) \sum_{j=1}^m \alpha_j(t) Q_m \phi_j dx dt.$$

Because $Q_m \phi_j \rightarrow 0 \text{ di } L^p(\Omega)$ for each j . Hence, we can prove that $P_m f(u_m)$ is convergent.

The facts imply each of the equation found in space $L^2(0, T; V) \cap L^p(\Omega_T)$ is convergent on those dual spaces. The dual space is $L^2(0, T; V^*) \cap L^q(\Omega_T)$. Equation

$$\frac{du}{dt} + Au = \chi. \quad (9)$$

Fulfil similarity in this room. To make more clear, providing the fact that

$$L^2(0, T; V^*) \cap L^q(\Omega_T) \subset L^q(0, T; H^{-s}).$$

Hence, it is absolute that Equation (9) fulfil similarity in that space.

Furthermore, we go on proving that $\chi = f(u)$. To do this, we use the weak convergence of u_m to u di $L^2(\Omega_T)$ and also Lemma 2. Since $u_m \rightarrow u \text{ di } L^2(\Omega_T)$, using the Lemma 1 to guarantee the existence of sub term u_{n_j} in such a way that $u_{n_j}(x, t) \rightarrow u(x, t)$ for each $(x, t) \in \Omega_T$.

So that by using the continuity of f , that $f(u_{n_j}(x, t)) \rightarrow f(u(x, t))$ for almost every $(x, t) \in \Omega_T$ along the boundary on $f(u_m) \text{ di } L^q(\Omega_T)$ provided on Equation (7), we apply Lemma 1 to conclude that $f(u_{n_j}(x, t)) \rightarrow f(u(x, t)) \text{ di } L^q(\Omega_T)$. From singularity of weak limits, cause $\chi = f(u)$.

To prove that solution $u(t)$ continue from $[0, T]$ on $L^2(\Omega)$, we investigate $u(0, T: V) \cap L^p(\Omega_T)$, using Theorem 1 to conclude $u \in C^0([0, T]: L^2(\Omega))$.

To prove $u(0) = u_0$, take $\phi \in C^1([0, T]; V) \cap L^p(\Omega)$, with $\phi(T) = 0$. Notice that, $\phi \in L^2(0, T: V) \cap L^p(\Omega_T)$. So that, by limiting equation

$$\left\langle \frac{du}{dt}, v \right\rangle + a(u, v) = \langle f(u), v \rangle + \langle \delta, v \rangle.$$

Implementing partial integral to variable t ,

$$\int_0^T -\langle u, \phi' \rangle + a(u, \phi) ds = \int_0^T \langle f(u(s)), \phi \rangle + \langle \delta, \phi \rangle ds + (u(0), \phi(0))$$

and applying Galerkin approach methods, we obtain

$$\int_0^T -\langle u_m, \phi' \rangle + a(u_m, \phi) ds = \int_0^T \langle P_m f(u_m(s)), \phi \rangle + \langle \delta, \phi \rangle ds + (u_0, \phi(0)).$$

Since $u_m(0) = P_m u_0 \rightarrow u_0$ we have $u(0) = u_0$.

In order to prove the singularity of solution, Let u_0 and v_0 be initial values on H . Then, form $w(t) = u(t) - v(t)$,

so that

$$\frac{dw}{dt} + Aw = f(u) - f(v), \quad w(0) = u_0 - v_0.$$

By multiplying it with w and integrating it along Ω , we obtain

$$\frac{1}{2} \frac{d}{dt} |w|^2 + \|w\|^2 = (f(u) - f(v), u - v).$$

It is worth noted that the boundary on f' is found in Equation (6). This prove that

$$\begin{aligned} (f(u) - f(v), u - v) &= \int_{\Omega} [f(u(x)) - f(v(x))](u(x) - v(x)) dx \\ (f(u) - f(v), u - v) &= \left(\int_{v(x)}^{u(x)} f'(s) ds \right) (u(x) - v(x)) dx \\ &\leq l |u(x) - v(x)|^2 dx \\ &\leq l |u - v|^2 \end{aligned}$$

hence

$$\frac{1}{2} \frac{d}{dt} |w|^2 \leq l |u - v|^2.$$

we then integrate to acquire

$$|u(x) - v(x)| \leq |u_0 - v_0| e^{lt}.$$

We prove the singularity if $u_0 = v_0$ an also depend on the continuity of the initial condition.

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